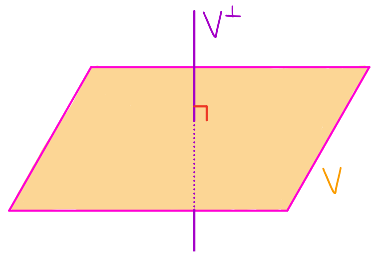
# Lecture 31. Orthogonal complements

**Def** Given a subspace $V$ of $\mathbb{R}^n$, its orthogonal complement $V^\perp$ is the set of all vectors in $\mathbb{R}^n$ which are orthogonal to all vectors in $V$.




**Note** $(V^\perp)^\perp = V$ (cf. $(A^T)^T = A$ for a matrix $A$)

**Thm** Given a matrix $A$, we have

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T) \text{ and } \mathrm{Nul}(A)^\perp = \mathrm{Col}(A^T) = \mathrm{Row}(A).$$

**pf** Let $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$ be the columns of $A$.

$\vec{v}$ lies in $\mathrm{Col}(A)^\perp$

$\iff$ $\vec{v}$ is orthogonal to $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$

$\iff$ $\vec{v}\cdot\vec{v}_1 = 0, \ \vec{v}\cdot\vec{v}_2 = 0, \cdots, \ \vec{v}\cdot\vec{v}_n = 0$

$\iff$ $A^T\vec{v} = \vec{0}$ ($A^T$ has rows $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n$)

$\iff$ $\vec{v}$ lies in $\mathrm{Nul}(A^T)$

Hence we have $\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$

For $A^T$, we find $\mathrm{Col}(A^T)^\perp = \mathrm{Nul}(A^{TT}) = \mathrm{Nul}(A)$

$\Longrightarrow$ $\mathrm{Nul}(A)^\perp = (\mathrm{Col}(A^T))^{\perp\perp} = \mathrm{Col}(A^T) = \mathrm{Row}(A)$

**Prop** Given a subspace $V$ of $\mathbb{R}^n$, we have $\dim(V) + \dim(V^\perp) = n$.

**pf** Take an $m\times n$ matrix $A$ whose rows form a basis of $V$

$\Longrightarrow$ $V = \mathrm{Row}(A)$ and $V^\perp = \mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$

$\Longrightarrow$ $\dim(V) + \dim(V^\perp) = n$ (Rank-nullity theorem)

Ex For each subspace of $\mathbb{R}^3$, find a basis of its orthogonal complement.

(1) The plane $2x+4y-3z=0$

Sol The plane is given by $\text{Nul}(A)$ with

$$A = \begin{bmatrix} 2 & 4 & -3 \end{bmatrix}.$$

The orthogonal complement is $\text{Nul}(A)^\perp = \text{Row}(A)$ $(= \text{Col}(A^T))$

Since $A$ has a unique row, $\text{Row}(A)$ has a basis given by $\boxed{\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}}$

row of A

Note In fact, the orthogonal complement of the plane $ax+by+cz=0$

is the line spanned by $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

(2) The intersection of the planes $x+y+z=0$ and $2x-3z=0$

Sol The space is given by $\text{Nul}(A)$ with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -3 \end{bmatrix}.$$

The orthogonal complement is $\text{Nul}(A)^\perp = \text{Row}(A)$ $(= \text{Col}(A^T))$

The two rows in $A$ are linearly independent.

(neither is a multiple of the other)

$\Rightarrow$ $\text{Row}(A)$ has a basis given by $\boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}}$

rows of A

(3) The line spanned by $\vec{v} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}$

Sol The line is given by the row space of

$$A = \begin{bmatrix} 2 & -2 & 4 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}$$

The orthogonal complement is $\mathrm{Row}(A)^{\perp} = \mathrm{Nul}(A)$

$$A\vec{x} = \vec{0} \Rightarrow x_1 - x_2 + 2x_3 = 0 \Rightarrow x_1 = x_2 - 2x_3 \underset{x_2 = s,\ x_3 = t}{\Rightarrow} \vec{x} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Hence Nul(A) has a basis given by $\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}}$

Note We can instead work with a column space using transpose.

(4) The plane spanned by $\vec{v} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 4 \\ 1 \\ 8 \end{bmatrix}$.

Sol The line is given by the row space of

$$A = \begin{bmatrix} 3 & 2 & -4 \\ 4 & 1 & 8 \end{bmatrix} \text{ with } \mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -8 \end{bmatrix}.$$

The orthogonal complement is $\mathrm{Row}(A)^{\perp} = \mathrm{Nul}(A)$

$$A\vec{x} = \vec{0} \Rightarrow \left\{ \begin{aligned} x_1 \quad + 4x_3 &= 0 \\ x_2 - 8x_3 &= 0 \end{aligned} \right. \Rightarrow \left\{ \begin{aligned} x_1 &= -4x_3 \\ x_2 &= 8x_3 \end{aligned} \right. \underset{x_3 = t}{\Rightarrow} \vec{x} = t\begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix}$$

Hence Nul(A) has a basis given by $\boxed{\begin{bmatrix} -4 \\ 8 \\ 1 \end{bmatrix}}$

Note This example is comparable to the last example in Lecture 29.

Ex Given a matrix $A$ with

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

find the dimension of each vector space.

(1) $\mathrm{Nul}(A)^\perp$

Sol $\mathrm{Nul}(A)^\perp = \mathrm{Row}(A)$ has dimension $\boxed{2}$ (number of leading 1s)

(2) $\mathrm{Col}(A)^\perp$

Sol $\dim(\mathrm{Col}(A)) = 2$ (number of leading 1s)

$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Col}(A)^\perp) = 3$ ($\mathrm{Col}(A)$ is a subspace of $\mathbb{R}^3$)

$\Longrightarrow \dim(\mathrm{Col}(A)^\perp) = 3 - \dim(\mathrm{Col}(A)) = 3 - 2 = \boxed{1}$

Note Since $\mathrm{Col}(A)^\perp$ is $\mathrm{Nul}(A^T)$ and not $\mathrm{Nul}(A)$, its dimension is not necessarily equal to the nullity of $A$. In fact, we have

$$\begin{cases} \dim(\mathrm{Col}(A)) + \dim(\mathrm{Col}(A)^\perp) = 3 & \text{(number of rows)} \\ \dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = 4 & \text{(number of columns)} \end{cases}$$

(3) $\mathrm{Row}(A)^\perp$

Sol $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$ has dimension $\boxed{2}$

(number of columns without a leading 1)